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Linear derivative Cartan formulation of general relativity

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Abstract. Beside diffeomorphism invariance also manifest $SO(3,1)$ local Lorentz invariance is implemented in a formulation of Einstein gravity (with or without cosmological term) in terms of initially completely independent vielbein and spin connection variables and auxiliary two-form fields. In the systematic study of all possible embeddings of Einstein gravity into that formulation with auxiliary fields, the introduction of a "bi-complex" algebra possesses crucial technical advantages. Certain components of the new two-form fields directly provide canonical momenta for spatial components of all Cartan variables, whereas the remaining ones act as Lagrange multipliers for a large number of constraints, some of which have been proposed already in different, less radical approaches. The time-like components of the Cartan variables play that role for the Lorentz constraints and others associated to the vierbein fields. Although also some ternary ones appear, we show that relations exist between these constraints, and how the Lagrange multipliers are to be determined to take care of second class ones. We believe that our formulation of standard Einstein gravity as a gauge theory with consistent local Poincaré algebra is superior to earlier similar attempts.

1 Introduction

For many decades the standard approach to the Hamiltonian formulation of general relativity (GR) has been the use of specific geometric variables, introduced by Arnowitt, Deser and Misner (ADM) [1]: the lapse and shift functions, the intrinsic curvature and the three space metric on space-like surfaces which determine the foliation corresponding to a time coordinate x^0. The Hamiltonian analysis leads to four constraints, the Hamiltonian and the diffeomorphism ones, which are all first class. At the quantum level the Hamiltonian constraint can by transcribed as a formal functional differential equation, the so-called Wheeler–DeWitt equation [2].

From the point of view of geometry it is well known that the description of a manifold by Cartan variables [3], (one-form) vierbeins e^A and the (one-form) spin connection ω^{AB} (with $A = 0, 1, 2, 3$, describing the local Lorentz coordinates), is the most comprehensive one. Only when the torsion vanishes, as it is assumed in Einstein's general relativity (GR), the spin connection is expressed in terms of the vierbeins and ceases to represent an independent variable. In this interpretation the metric $g = \eta^{AB} e_A \otimes e_B$ ($\mathrm{diag}\,\eta = (-1, 1, 1, 1)$) is not a fundamental, but rather a derived quantity. It owes its existence to the appearance of just this combination of vierbeins in a Lagrangian of a scalar test particle.

The opposite interpretation consists in taking the spin connection as the more basic variable.[1] In that way a closer resemblance to the well-studied Yang–Mills theories could be achieved. This has been the key to Ashtekar's identification of an (anti-) selfdual spin connection with an $SO(3)$ Yang–Mills field [5] which by introducing notions from the Yang–Mills case like Wilson loops allowed for many important new developments (cf. e.g. [6]). The price to pay in Ashtekar's original formulation was the extension to complex fields which, in the end, had to be reduced by imposing a reality condition. Including a second $SO(3)$ connection proportional to the Immirzi parameter β, Barbero [7] showed that most of the advantages of Ashtekar's formulation (e.g. polynomial constraints) could be retained for a real value of β which in Ashtekar's approach would have to be imaginary. However, that parameter introduced an ambiguity ("Immirzi ambiguity") in a quantized theory [8]. At present perhaps the most elaborate version of this general line of research is the one pursued up to the level of quantum theory by Thiemann [9].

Still, also in the developments originating from the connection type formulation the vanishing torsion condition is incorporated as a given relation between connection and vierbeins from the start.

On the other hand, the idea to at least initially treat both types of variables as independent ones goes back to

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[1] The most consequent "connection" formulation of gravity is perhaps the one by Capovilla and Jacobson [4].

Palatini [10][2] who showed it to be a peculiar property of the Hilbert–Einstein action for GR that it also yields the metricity condition and the condition of vanishing torsion. In its modern version [11] in terms of Cartan variables the "Hilbert–Palatini" action $\left(\epsilon_{0123} = -\epsilon^{0123} = 1\right)^3$,

$$S_{(\mathrm{HP})} = \frac{1}{2} \int\limits_{\mathcal{M}_4} R^{AB} \wedge e^C \wedge e^D \epsilon_{ABCD} \ , \tag{1}$$

with the curvature two-form

$$R^{AB} = d\omega^{AB} + \omega^A{}_C \wedge \omega^{CB} \ , \tag{2}$$

by independent variation of the vierbeins e_A and of the spin connection ω_{AB} fields yields vanishing torsion,

$$\tau^A = (De)^A = de^A + \omega^A_B \wedge e^B = 0, \tag{3}$$

and the Einstein equations of GR. Because (1) only contains first derivatives of ω^{AB}, this is usually referred to as the first order formulation of GR.

The connection form can be regarded as an $SO(3,1)$ gauge field. This implies that ω^{AB} defines a connection in a principal fiber bundle $\mathcal{P}\ (\mathcal{M}, SO(3,1))$ or vector bundles associated with it. However, there is a basic difference from ordinary gauge theories: the local gauge symmetry is not independent of the effective action on the manifold; it is rather the same. Nevertheless, the interpretation as a gauge field turns out to be more successful when ω is treated as an independent field.

When the description of geometry by Cartan variables, instead of the metric as in the original ADM formalism, is taken seriously it is natural to include Lorentz constraints Ω^{AB} which satisfy the Poisson brackets appropriate for generators of local Lorentz invariance. Also the spatial vierbein components $e^A{}_i$ $(i = 1, 2, 3)$ must be associated to conjugate momenta π_e^{Ai} which either from an action like (1) become constraints by themselves or must be expressible in terms of other dynamical variables. On the other hand, there were good reasons to retain the successful ADM formulation in terms of lapse and shift. This program of "tetrad gravity" started with a seminal paper of Deser and Isham [12] and has been elaborated by several authors (exemplary references are [13]). A common complication of approaches of this type of formulations has been the complexity of the computations required for the determination of the Poisson brackets.[4]

[2] More precisely, he viewed the metric and the covariant derivative as independent variables in the Hilbert–Einstein action.

[3] This follows directly by interpreting ϵ as volume form and following the usual index convention $f_A := f(E_A)$ for indices $A, B \ldots$ from the begining of the alphabet, referring to local (anholonomic) Lorentz coordinates $A = (\underline{0}, \underline{1}, \underline{2}, \underline{3}) = (\underline{0}, a)$. Indices $I = (0, 1, 2, 3) = (0, i)$ from the middle of the alphabet are related to holonomic coordinates. In the action the prefactor, depending on Newton's constant, is normalized to one.

[4] A shortcut proposed by Teitelboim [14], namely to rely on symmetry relations to avoid certain calculations, turned out to be not applicable when terms of higher than linear order in the constraints appeared [15].

To the best of our knowledge the first to consider tetrad gravity in a first order formulation, e.g. with *both* e_A and ω_{AB} as independent fields in the action (1), were the authors of [16]. The condition of vanishing torsion appears as a second class constraint which expresses the spin connection in terms of the vierbeins and necessitates the introduction of Dirac brackets.

As mentioned already above, with the advent of Ashtekar's variables [5] the emphasis shifted towards the connection components ω_{ABI} of $\omega_{AB} = \omega_{ABI}\, dx^I$. In the Palatini-type action (1) the corresponding conjugate momenta $\pi_\omega^{(AB)}$ become bi-vector[5] components proportional to $(e)^{AB} = e^A \wedge e^B$. This can be taken care of by adding the constraints, $\phi^{ij} = \epsilon_{ABCD} \pi_\omega^{ABi} \pi_\omega^{CDj} \approx 0$, so that the spatial vierbein variables are eliminated from the start (cf. e.g. [17]). A similar elimination of the vierbeins is the one of [4] where only the connection is kept as a variable already at the Lagrangian level.

The philosophy of our present paper is rather unconventional in the sense that *both* Cartan variables are treated "democratically"; vanishing torsion does not appear by the Palatini mechanism, but is imposed explicitly with the help of a Lagrangian multiplier which at first sight may appear as something of an overkill. However, it owes its basic idea to progress made recently in $1+1$ dimensional dilaton gravity theories, which includes spherically reduced gravity from d dimensions, but also the string-inspired dilaton model of CGHS [18] and other simpler models, the most prominent being the one of Jackiw and Teitelboim [19]. One of the present authors (W.K.) [20] together with Schwarz at first in a special model with non-vanishing torsion [21] realized the importance of using a temporal gauge in the Cartan variables which led to extremely simple treatments of those models in the classical and quantum case [22,23]. In this development the formulation as the linear derivative action of a "Poisson Sigma model" [24,23]

$$S_{(\mathrm{LD})} = \int\limits_{\mathcal{M}_2} \left[X d\omega + X_a\, De^a \ + \epsilon \mathcal{V} \left(X^a X_a, X \right) \right] \tag{4}$$

with real auxiliary fields X, X^a ($a = 0, 1$; the Hodge star of $2d\omega$ becomes R, the Ricci scalar in $d = 2$; $\epsilon = \frac{1}{2} \epsilon_{ab}\, e^a \wedge e^b$ is the volume form) played an important role. Indeed all physically interesting models could be covered by a "potential" $\mathcal{V}$ quadratic in X^a, i.e. generically including non-vanishing torsion. By eliminating (algebraically) X_a and the torsion dependent part of the spin connection $\omega_{ab} = \omega \epsilon_{ab}$ the usual torsionless dilaton gravity action with dilaton field X could be reproduced, the two formulations being locally and globally equivalent at the classical as well as at the quantum level [22,25]. However, on the basis of (4) even a background independent quantization of spherically reduced (torsionless Einstein) gravity has been possible which actually reduces to (local) quantum triviality in the absence of matter [22,23]. This formulation recently has turned out to be able to solve as well

[5] More on bi-vectors can be found in Appendix A.

some long-standing problems in $N = (1,1)$ dilaton supergravity theories [26]. With respect to applications in string theory the first determination of the full action (including fermionic fields) for $N = (2,2)$ supergravity also has been a consequence of this approach [27].

Actually the unique mathematical properties of an action like (4) may be traced to the fact that it represents a (gravitational) example of a Poisson Sigma model [24]. Such models share an important feature with Yang–Mills theory, namely that the algebra of Hamiltonian (secondary, first class) constraints closes without derivative terms on δ-functions, a property which lies at the root of the relatively easy tractability of gravity theories in two dimensions.

Thus it appears natural to generalize (4) from $\mathcal{M}_2$ to a four dimensional manifold $\mathcal{M}_4$ with two-form auxiliary fields X^{AB}, Y^A replacing the zero-forms X, X^a, and, of course, a polynomial in $e^A \wedge e^B$ and X^{AB} instead of the potential term $\epsilon\mathcal{V}$. This generalization even may possess physical justification. Remembering that in the $1+1$ dimensional case one of the auxiliary fields (X in (4)) in spherically reduced GR owes its presence to the dilaton field, the same fact could be reflected in certain parts of the new X^{AB} which could be dilaton fields, related to some compactification mechanism from a gravity theory with dimension larger than four.

In the diploma thesis of one of the present authors (H.S.) [28] this idea passed a positive test: an action of generic type (4) on $\mathcal{M}_4$ with a large number of free parameters was shown to lead – up to a topological term – to Einstein (–de Sitter) gravity, i.e. the action $S_{(\mathrm{HP})}$ of (1), when the fields X^{AB}, Y^A are eliminated by algebraic equations of motion (EOM). In this preliminary work many questions were left open: a systematic analysis of all possibly relevant topological terms, a detailed comparison with previous formulations, which, in principle, should be contained as special cases, a comparison of EOM etc. Therefore, a detailed confrontation of a general Palatini-type action with the new formulation is the subject of Sect. 2, respectively Sect. 3.

In the canonical analysis (Sect. 4) we have to treat a rather large set of constraints, because, in contrast to $d = 2$, only *part* of the components of X^{AB} and Y^A become canonically conjugate momenta, the rest being Lagrange multipliers, which turn out to be completely determined.[6] The structural similarity to a gauge theory suggests a foliation directly in our "time" coordinate x^0 *without* the introduction of lapse and shift variables. This represents perhaps the most basic difference with respect to the usual tetrad gravity, where almost everywhere[7] the ADM decomposition had been used. In our case the resulting constraints are polynomial and their Poisson brackets (with one "inessential" exception) rather yield delta functions on the right-hand side, just as for a non-abelian gauge theory and the aforementioned Poisson Sigma models – and not derivatives thereof as for the ADM constraints. As we proceed in this analysis we have opportunities to compare with previous works and their relation to our present approach. Our analysis suggests that a linear combination of the initial constraints yields first class ones that fulfil the Poincare algebra of special relativity as algebra of the Poisson brackets.

In the final section, Sect. 5, we summarize the results obtained, list open problems and indicate some possible directions of future work. Some calculational details are relegated to the appendices: important formulas on vector forms are collected in Appendix A, the rules of the "hat calculus" are listed in Appendix B. Some complicated Poisson brackets are collected in Appendix C. All brackets have been computed by hand and checked by a computer algebra package. In Appendix D we give details on the determination of the Lagrange multipliers whose final results appear in the main text.

2 Alternative formulations of Einstein gravity

In order to create a basis for the comparison of the actions in the ensuing EOM of our approach in terms of Cartan variables and auxiliary fields (Sect. 3) we collect here some well-known formulations in an appropriate notation.

First we note that $S_{(\mathrm{HP})}$ of (1) can be written in several equivalent ways. Introducing the duality operation for the two-form components F^{AB} of a bi-vector F,[8]

$$\widehat{F_{AB}} := \frac{1}{2}\,\epsilon_{ABCD}\,F^{CD}, \tag{5}$$

the identity

$$\widehat{e^{AB}} = *e^{AB}\,, \tag{6}$$

with $f^{AB} := \left(f^2\right)^{AB} = (f \wedge f)^{AB} = f^A \wedge f^B = -f^{BA}$, for an oriented orthonormal co-basis e^A can be derived easily. Equation (6) allows one to replace the hat by the Hodge star operation in (1),

$$S_{(\mathrm{HP})} = \int \widehat{e^{AB}} \wedge R_{AB} = \int *e^{AB} \wedge R_{AB}\,. \tag{7}$$

Hence, in terms of this type one can move not only the hat, but also the Hodge star operation freely onto R_{AB}, the anholonomic components of the Ricci tensor, as well. It will be important in the next section that all such terms are equivalent.

Variation of $S = S_{(\mathrm{HP})} + S_{(\mathrm{M})}$, where $S_{(\mathrm{M})}$ is some matter action, with respect to δe^A yields[9] the Einstein equations in the form

$$\widehat{R_{BA}} \wedge e^B = \left(R_{AC} - \frac{1}{2}\,R\,\eta_{AC}\right) \wedge \left(*e^C\right) = \frac{1}{2}T_A\,, \tag{8}$$

[6] This must be the case, of course, because we do not introduce new physical degrees of freedom.

[7] Among more recent exceptions we are only aware of [29].

[8] Appendices A and B should be consulted for more details on bi-vectors, respectively the hat calculus.

[9] Here and thereafter we neglect surface terms.

where the energy momentum three-form T_A is defined in terms of a matter action $S_{(\mathrm{M})}$ as

$$\delta S_{(\mathrm{M})} = \int\limits_{\mathcal{M}_4} \left(\delta e_A \wedge T^A + \frac{1}{2} \delta\omega_{AB} \wedge S^{AB} \right) . \tag{9}$$

Variation of $\delta\omega$ leads to

$$\left(De^2\right)^{AB} = \tau^A \wedge e^B - \tau^B \wedge e^A = \frac{1}{2} S^{AB} , \tag{10}$$

with the torsion two-form (cf. (3))

$$\tau^A = de^A + \omega^A_{\ B} \wedge e^B . \tag{11}$$

For vanishing "spin current" S^{AB} it can be shown that the vanishing of the left-hand side of (10) holds if and only if the torsion τ^A vanishes. This is a special case of a simple lemma (see Appendix A) which, in a less trivial application, also allows one to determine the solution τ^A in terms of a non-vanishing S^{AB} in (10).

The first Bianchi identity $(\mathrm{D}\tau)^A =: \mathcal{D}^{A\ B}\tau_B = R^{A\ B} \wedge e_B$ guarantees that adding an analogous "hat-less" term to (7) with parameter $\gamma \in \mathbb{R}$

$$S'_{(\mathrm{HP})} = \int\limits_{\mathcal{M}_4} \left(\widehat{e}^{AB} \wedge R_{AB} + \gamma e^{AB} \wedge R_{AB} \right) , \tag{12}$$

the EOM (8) are modified to

$$\widehat{R_{AB}} \wedge e^B = \frac{1}{2} T_A - \gamma \left(\mathrm{D}\tau\right)_A . \tag{13}$$

The "torsion equation" (10) also changes into

$$\left(\mathrm{D}e^2\right)^{AB} = \frac{1}{2\gamma^2} \left(S^{AB} + \gamma \widehat{S^{AB}} \right) . \tag{14}$$

Thus, due to the first Bianchi identity also in that case the Einstein equations are recovered for vanishing spin current S^{AB} and vanishing torsion. Identifying γ^{-1} with the Immirzi parameter β shows that Barbero's Hamiltonian [7] can be reproduced in this way. Although (7) and (12) yield the same classical solutions to the EOM, at the quantum level, of course, the theories would be expected to be different.[10]

Further terms which could be added to $S_{(\mathrm{HP})}$ without changing the EOM are two topological ones: the Gauss–Bonnet–Chern form

$$L_{(\mathrm{GBC})} = \frac{1}{16\pi^2} \widehat{R_{AB}} \wedge R^{AB} \tag{15}$$

and the Pontrijagin form

$$L_{(\mathrm{P})} = \frac{1}{8\pi^2} R_{AB} \wedge R^{AB} . \tag{16}$$

[10] We use this cautious formulation because according to the work of [30,31] there exists an Ashtekar-like formulation where the connection is the one of the full $SO(3,1)$ and where γ appears as an "anomaly candidate", to be removed by renormalization.

By analyzing all terms quadratic in R_{AB} according to the rules of Appendix B (with hats and stars distributed over both factors) it can be verified by means of the second Bianchi identity $(\mathrm{D}R)^{AB} = \left(\widehat{\mathrm{D}R^{AB}}\right) = 0$ that (15) and (16) are the only possibilities. Thus the most general Palatini-type action to be compared with our own formulation below can be written as

$$\begin{aligned} S^{(\mathrm{ext})}_{(\mathrm{HP})} &= \int\limits_{\mathcal{M}_4} \Big(R_{AB} \wedge \widehat{e^{AB}} + \gamma R_{AB} \wedge e^{AB} + \rho R_{AB} \wedge R^{AB} \\ &\quad + \sigma R_{AB} \wedge \widehat{R^{AB}} + \frac{\Lambda}{12} \widehat{e^{AB}} \wedge e_{AB} \Big) , \end{aligned} \tag{17}$$

with general parameters γ, ρ, σ. The last term with the cosmological constant Λ allows one to include the Einstein–de Sitter case as well.

For completeness we also place the complex formulation of real GR [5] into the present context. For the (anti-) selfdual connections

$$\omega^{\pm}_{AB} := \frac{1}{2} \left(\omega_{AB} \pm \mathrm{i}\widehat{\omega_{AB}} \right) \tag{18}$$

only three of the six components of $\omega^{\pm}_{AB}$ are linearly independent over $\mathbb{C}$. Analogously, the (anti-) selfdual curvature is defined by

$$R^{\pm}_{AB}(\omega) := \frac{1}{2} \left(R_{AB}(\omega) \pm \mathrm{i}\widehat{R_{AB}}(\omega) \right) . \tag{19}$$

Applying the hat operation (5) to (19) the HP action (1) can be rewritten as

$$S_{(\mathrm{HP})} = \mathrm{i} \int\limits_{\mathcal{M}_4} \left[-R^{+}_{AB} + R^{-}_{AB} \right] \wedge e^{AB} . \tag{20}$$

Choosing $\omega^{\pm}_{0a}$ as a basis ($A = 0, a$ etc.) from (18) and (19) the components of the (anti-) selfdual curvature can be expressed as

$$R^{\pm}_{0a} = d\omega^{\pm}_{0a} \pm \mathrm{i}\epsilon_{0a}{}^{cd} \omega^{\pm}_{0c} \wedge \omega_{0d}^{\ \pm} , \tag{21}$$

with the structure constants $\epsilon_{0a}{}^{cd}$ implying that the Lie group for $\omega^{\pm}_{0a}$ is $SO(3,\mathbb{C})$. Ashtekar's variables arise if one of the two terms in the real action (20) is dropped so that the contribution of the variation with respect to the "complex conjugate" connection one-form disappears. In contrast ours will be a full $SO(3,1)$ gauge formulation as shown below.

3 Actions with auxiliary fields

3.1 Equivalent action

Motivated by the success of the analogous $1 + 1$ dimensional dilaton theory (4) the basic ansatz of our approach

$$S_{(\mathrm{LD})} = S_{(X)} + S_{(Y)} + S_{(\tilde{\Lambda})} , \tag{22}$$

with

$$S_{(X)}(X,\omega,e) = \int_{\mathcal{M}} \frac{1}{2}\Big(a_0 X^{AB}\wedge X_{AB} + a_1 X^{AB}\wedge *X_{AB} + a_2 X^{AB}\wedge \widehat{X_{AB}} + a_3 X^{AB}\wedge *\widehat{X_{AB}} + b_0 X^{AB}\wedge R_{AB} + b_1 X^{AB}\wedge *R_{AB} + b_2 X^{AB}\wedge \widehat{R_{AB}} + b_3 X^{AB}\wedge *\widehat{R_{AB}} + c_0 X^{AB}\wedge e_{AB} + c_2 X^{AB}\wedge \widehat{e_{AB}}\Big), \tag{23}$$

$$S_{(Y)} = \int_{\mathcal{M}} Y^A \wedge (\mathrm{D}e)_A\ , \tag{24}$$

$$S_{(\tilde{\Lambda})} = \frac{\widetilde{\Lambda}}{12}\int_{\mathcal{M}} \widehat{e^{AB}}\wedge e_{AB}\,, \tag{25}$$

consists of at most quadratic expressions in auxiliary two-form fields X^{AB}, Y^A, the curvature R^{AB} and the bi-vector e^{AB}. Equation (24) explicitly imposes the condition of vanishing torsion, although, as is well known (cf. also Sect. 2), the Palatini mechanism implies this anyhow. However, the advantage is that in this way, at least at the start, an independent momentum variable, canonical conjugate to e_{Ai}, has been introduced. The eventual appearance of a cosmological term has been foreseen by the inclusion of $S_{(\tilde{\Lambda})}$ in (25). It should be noted, however, that the "effective" cosmological constant Λ in the equivalent Einstein–de Sitter theory will also acquire contributions from combinations of the constants a_i, b_i $(i = 1, ..., 4)$, c_0, c_2. The restriction to coefficients c_0 and c_2 only in the last two terms of (23) originates from the identity (6). In the rest of (23) simply all independent starry and hatted terms have been collected. It should be noted that the formulations of gravity as a BF-theory exhibit a certain similarity to our ansatz, the main difference being that our two-form auxiliary fields X^{AB} are not simply proportional to e^{AB} (cf. (29) below) [32] and Y^A (at least initially) does not vanish. A more compact form of (23) is obtained by noting that the hat as well as the Hodge star operator – in our applications to bi-vectors only – act like (mutually commuting) imaginary units of two independent complex structures ($\mathbf{i}^2 := -1, \mathbf{j}^2 := -1, \mathbf{ij} = \mathbf{ji}, (a+\mathbf{i}b)^* := a - \mathbf{i}b, (c+\mathbf{j}d)^\wedge := c - \mathbf{j}d$)

$$\begin{aligned} a * X_{AB} &=: a\mathbf{i}X_{AB}, \\ a\,\widehat{X_{AB}} &=: a\mathbf{j}X_{AB}, \\ a * \widehat{X_{AB}} &=: a\mathbf{ij}X_{AB}. \end{aligned} \tag{26}$$

This permits abbreviations of the form

$$\mathbf{a} = a_0 + a_1\mathbf{i} + \mathbf{j}(a_2 + \mathbf{i}a_3) =: a_{(\mathbf{1})} + \mathbf{j}a_{(\mathbf{2})} \tag{27}$$

in (23) which may be considered as elements (27) of a "bi-complex" algebra $\mathbb{C}^2$: with non-trivial, i.e. different from zero, not necessarily invertible elements $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (for a first analysis cf. [28]):

$$S_{(X)} = \int \frac{1}{2}\Big((\mathbf{a}X)^{AB}\wedge X_{AB} + (\mathbf{b}X)^{AB}\wedge R_{AB} + (\mathbf{c}X)^{AB}\wedge e_{AB}\Big). \tag{28}$$

The resulting EOM from variation of δX_{AB}

$$X^{AB} = -\frac{\mathbf{a}^{-1}}{2}\left(\mathbf{b}R^{AB} + \mathbf{c}e^{AB}\right) \tag{29}$$

are solved with the inverse of (27),

$$\mathbf{a}^{-1} = \frac{a_{(\mathbf{1})} - \mathbf{j}a_{(\mathbf{2})}}{{a_{(\mathbf{1})}}^2 + {a_{(\mathbf{2})}}^2} = |a|^{-4}\left({a_{(\mathbf{1})}}^2 + {a_{(\mathbf{2})}}^2\right)^* \mathbf{a}^\wedge\ , \tag{30}$$

where

$$\begin{aligned} a_{(\mathbf{1})} &:= a_0 + \mathbf{i}a_1, \\ a_{(\mathbf{2})} &:= a_2 + \mathbf{i}a_3, \\ |a|^4 &:= \left({a_{(\mathbf{1})}}^2 + {a_{(\mathbf{2})}}^2\right)\left({a_{(\mathbf{1})}}^2 + {a_{(\mathbf{2})}}^2\right)^* \geq 0. \end{aligned} \tag{31}$$

The existence of (30) and hence of (29) is guaranteed if the a_i's are chosen such that $|a| \neq 0$.

The linearity of (29) guarantees that the variational principle remains unchanged by the insertion of (29) in (28). Thus the extended HP-action (17) is reproduced if after insertion of (29) the coefficients in

$$S_{(X)} = \int_{\mathcal{M}} \frac{1}{2}\left[(\mathbf{k}R_{AB})\wedge R^{AB} + (\mathbf{l}R_{AB})\wedge e^{AB} + (\mathbf{m}e_{AB})\wedge e^{AB}\right] \tag{32}$$

are chosen as

$$\begin{aligned} \mathbf{k} &= -\frac{1}{4}\mathbf{a}^{-1}\mathbf{b}^2 = 2\,(\rho + \mathbf{j}\sigma)\ , \\ \mathbf{l} &= -\frac{1}{2}\mathbf{a}^{-1}\mathbf{bc} = 2\,(\gamma + \mathbf{j})\ , \end{aligned} \tag{33}$$

$$\mathbf{m} = -\frac{\mathbf{a}^{-1}}{4}\mathbf{c}^2 = m_0 + \mathbf{j}m_2\,. \tag{34}$$

In the ansätze (33) for the RHS expressions involving the second "imaginary" unit $\mathbf{i}$ do not appear. The reason is that there are no topological terms of the curvature two-form R^{AB} containing the Hodge star in (17). This implies that the bi-complex algebra $\mathbb{C}^2$ may be effectively projected upon $\mathbb{C}$ (cf. the property of the full solution, (38) below).

With the ansatz (34) for $\mathbf{m}$ the identity

$$4\mathbf{km} = \mathbf{l}^2 \tag{35}$$

directly determines

$$m_2 = c_0 + c_2\,\gamma = \frac{2\rho\gamma + \sigma\left(1-\gamma^2\right)}{2\left(\sigma^2+\rho^2\right)}. \tag{36}$$

In (34) m_0 is irrelevant because $e_{AB}\wedge e^{AB} \equiv 0$, whereas (35) together with (25) provide contributions to the cosmological constant Λ in (17):

$$\Lambda = \widetilde{\Lambda} + 6m_2. \tag{37}$$

The identity (35) implies that the coefficients $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in the action (28) cannot be obtained uniquely from (33) and (34). Inserting the full solution with $\mathbf{q} \in \mathbb{C}^2$,

$$\mathbf{a} = -4\mathbf{q}^2\mathbf{k}, \ \mathbf{b} = -4\mathbf{qk}, \ \mathbf{c} = -2\mathbf{ql}, \tag{38}$$

into the action $S_{(X)}$ of (23) shows that the arbitrary invertible element $\mathbf{q}$ corresponds to a rescaling $\mathbf{q}X^{AB} \to X^{AB}$ which is always possible.[11] If the coefficient $\mathbf{b}$ of the term $X_{AB} \wedge R^{AB}$ is chosen to be 1 as in the 2D counterpart (4) we may fix the overall factor as $-4\mathbf{kq} = \mathbf{b} = 1$. Therefore, in

$$S_{(X)} = \int\limits_{\mathcal{M}} \frac{1}{2} X_{AB} \wedge \left(R^{AB} + \mathbf{a}X^{AB} + \mathbf{c}e^{AB}\right) \tag{39}$$

the coefficients

$$\mathbf{a} = -\frac{\rho - \mathbf{j}\sigma}{8\left(\rho^2+\sigma^2\right)}, \ \mathbf{c} = \frac{\rho\gamma + \sigma + \mathbf{j}\left(\rho - \sigma\gamma\right)}{2\left(\rho^2+\sigma^2\right)} \tag{40}$$

together with (36) provide agreement with the generalized Palatini-type action (17). As at least one of the coefficients ρ or σ must be non-vanishing the appearance of at least one topological term (15) or (16) is mandatory. The coefficient $\mathbf{c}$ cannot be made to disappear altogether. Not surprisingly, also the terms quadratic in X must be present always. A solution with $\mathbf{b} = 0$ is not possible (cf. (33)). It would correspond to the BF-model of [32]. Our present formulation (39) can be interpreted as the most comprehensive version of a generalized BF-model [33]. On the other hand, dropping the term with the Immirzi parameter ($\gamma = 0$) in (12) according to (40) is perfectly consistent with the ansatz (22), respectively (28).

An alternative nice parametrization of (40) follows from $\rho = r\cos\varphi$, $\sigma = r\sin\varphi$:

$$\mathbf{a} = -(8r)^{-1}\mathrm{e}^{-\mathbf{j}\varphi}, \ \mathbf{c} = (2r)^{-1}\,\mathrm{e}^{-\mathbf{j}\varphi}\,(\gamma + \mathbf{j}), \tag{41}$$

which interpolates by variation of φ between vanishing ρ, respectively σ. It also shows that the most "symmetric" solution for $\mathbf{a}$ and $\mathbf{c}$ is the one for $\gamma = 0$.

3.2 Equations of motion from auxiliary field action

The algebraic equivalence of the actions (39) and (17) – even by only linear relations – guarantees identical dynamical content. Nevertheless, the explicit form of the EOM from (39) will be very useful for comparison with the ones derived from the Hamiltonian. Variations of δY_A, δX_{AB}, $\delta\omega_{AB}$ and δe_A, respectively yield

$$\tau^A = 0, \tag{42}$$

$$2\mathbf{a}X^{AB} + R^{AB} + \mathbf{c}e^{AB} = 0, \tag{43}$$

$$(DX)^{AB} + \left(Y^A \wedge e^B - Y^B \wedge e^A\right) = 0, \tag{44}$$

$$(DY)^A + \mathbf{c}X^{AB} \wedge e_B + \frac{\widetilde{\Lambda}}{3!}\epsilon^A{}_{BCD}e^B \wedge e^{CD} = 0, \tag{45}$$

which allow for further simplification. The commutativity of the hat operation with the covariant derivative applied to (43) leads to

$$2\mathbf{a}\,(DX)^{AB} + (DR)^{AB} + \mathbf{c}\left(De^2\right)^{AB} = 0, \tag{46}$$

where the second term vanishes by Bianchi's second identity. The third one is eliminated by (10) and (42). In (44) this leads to the disappearance of the first term, and the rest by the same argument as for (10) $\left(S^{AB} = 0 \Leftrightarrow \tau^A = 0\right)$ forces $Y^A = 0$.

Therefore, the last two equations (44) and (45) may be replaced by

$$Y^A = 0\,, \tag{47}$$

$$\mathbf{c}X^{AB} \wedge e_B + \frac{\widetilde{\Lambda}}{6}\epsilon^A{}_{BCD}e^B \wedge e^{CD} = 0. \tag{48}$$

It is straightforward to verify that the set (37), (42), (43), (47) and (48) is equivalent to the EOM from (17). We also emphasize that the vanishing of torsion by (42) as well as of the related auxiliary field Y^A, according to (47), in our approach occurs on-shell only.

4 Hamiltonian analysis

4.1 Canonical Hamiltonian

The action (39) is of Hamiltonian form. Due to the linearity in the derivatives the momenta are directly associated to (part of the components of) the auxiliary fields X^{AB}, Y^A. The identification in such cases is an allowed short cut (cf. [28]), although, strictly speaking the corresponding relations are second class constraints.

Using the transcription rule in the typical two-form multiplications

$$U \wedge V = \left(U_{0i}\,V_{jk} + U_{jk}V_{0i}\right)\frac{\epsilon^{ijk}}{2}d^4x \tag{49}$$

for $\epsilon^{0ijk} = -\epsilon^{ijk}$ (the Levi-Civitá symbol with $\epsilon^{123} = 1$) to obtain the different pieces of the action (39) in terms of corresponding parts of the Lagrangian density $\mathcal{L} = \mathcal{L}_{(X)} + \mathcal{L}_{(Y)} + \mathcal{L}_{(\tilde{\Lambda})}$, the result is

$$\begin{aligned}\mathcal{L}_{(X)} = \frac{\epsilon^{ijk}}{4}\Big\{&X^{AB}{}_{0i}\left(2\mathbf{a}X_{ABjk} + R_{ABjk} + \mathbf{c}e_{ABjk}\right)\\ &+ X^{AB}{}_{jk}\left(\partial_0\omega_{ABi}\right) + e_{AB0i}\mathbf{c}X^{AB}{}_{jk}\\ &+ \omega_{AB0}\left(\partial_i X^{AB}{}_{jk} + \omega^A_{Ci}X^{CB}{}_{jk} - \omega^B{}_{Ci}X^{CA}{}_{jk}\right)\Big\}\,,\end{aligned} \tag{50}$$

$$\begin{aligned}\mathcal{L}_{(Y)} = \frac{\epsilon^{ijk}}{2}\Big\{&Y^A{}_{0i}\tau_{Ajk} + \left(\partial_0 e_{Ai}\right)Y^A{}_{jk}\\ &+ e_{A0}\left(\partial_i Y^A{}_{jk} + \omega^A{}_{Ci}\,Y^C{}_{jk}\right)\end{aligned} \tag{51}$$

[11] This possibility removes the only variable which could be an element of the whole bi-complex algebra. Nevertheless, the latter is useful to keep the discussion very simple but still completely general, at the intermediate steps leading to (39) and (40), but also in the Hamiltonian analysis below.

$$+\ \frac{1}{2}\omega_{AB0}\left(Y^{A}{}_{jk}e^{B}{}_{i}-Y^{B}{}_{jk}\,e^{A}{}_{i}\right)\Big\}\,,$$

$$\mathcal{L}_{(\tilde{\Lambda})}=\epsilon^{ijk}\frac{\widetilde{\Lambda}}{6}\,e_{A0}\epsilon^{A}{}_{BCD}\,e^{B}{}_{i}\,e^{C}{}_{j}\,e^{D}{}_{k}. \tag{52}$$

In (50) and (51) as in all previous computations total divergencies again have been disregarded.

The momenta conjugate to ω_{ABi}, respectively e_{Ai}, with x^0 defined as a time coordinate for any coordinate map $\left(x^I\right)$, can be read off from (50) and (51):

$$\pi_\omega^{ABi}=\Pi^{ABi}:=\frac{\epsilon^{ijk}}{2}X^{AB}{}_{jk}\,,\ A<B\,, \tag{53}$$

$$\pi_e^{Ai}=\pi^{Ai}:=\frac{\epsilon^{ijk}}{2}Y^{A}{}_{jk}\,, \tag{54}$$

whereas the conjugate momenta of ω_{AB0} and e_{A0} vanish as primary constraints. Therefore, we may effectively drop the set $e_{A0},\omega_{AB0},X_{AB0i},Y_{A0i}$ from the ranks of canonical variables, but treat them as Lagrange multipliers. This reduces the dimension of our phase space considerably from $N_{\rm ph}=2\cdot(16+24+36+24)=200$ to $N_{\rm ph}=2\cdot(12+18)=60$. The canonical Hamiltonian (from now on we use $\partial_0 f=:\dot{f}$, $x_{ABi}:=X_{AB0i}$, $y_{Ai}:=Y_{A0i}$)

$$\begin{aligned}H_{(\mathrm{can})}&=\int \mathrm{d}^3x\left(\frac{1}{2}\Pi^{ABi}\,\dot{\omega}_{ABi}+\pi^{Ai}\,\dot{e}_{Ai}-\mathcal{L}\right)\\&=\int \mathrm{d}^3x\left(e_{A0}E^{A}+\ \frac{1}{2}\omega_{AB0}\Omega^{AB}\right.\\&\qquad\left.+\frac{1}{2}x_{ABi}C^{ABi}+y_{Ai}T^{Ai}\right)\end{aligned} \tag{55}$$

depends on the constraints

$$\Omega^{AB}=\Omega^{AB}_{(X)}+\Omega^{AB}_{(Y)}\approx 0\,, \tag{56}$$

$$E^{A}=E^{A}_{(X)}+E^{A}_{(Y)}+E^{A}_{(\tilde{\Lambda})}\approx 0\,, \tag{57}$$

where in all cases the origin of the respective contributions

$$-\Omega^{AB}_{(X)}=\mathcal{D}_i\Pi^{ABi}=\partial_i\Pi^{ABi}+\omega^{A}{}_{Ci}\Pi^{CBi}-\omega^{B}{}_{Ci}\Pi^{CAi}\,, \tag{58}$$

$$-\Omega^{AB}_{(Y)}=\pi^{Ai}e^{B}{}_{i}-\pi^{Bi}e^{A}{}_{i}\,, \tag{59}$$

$$-E^{A}_{(X)}=\mathbf{c}\Pi^{ABi}e_{Bi}\,, \tag{60}$$

$$-E^{A}_{(Y)}=\mathcal{D}_i\pi^{Ai}=\partial_i\pi^{Ai}+\omega^{A}{}_{Bi}\pi^{Bi}\,, \tag{61}$$

$$-E^{A}_{(\tilde{\Lambda})}=\frac{\widetilde{\Lambda}}{6}\epsilon^{ABCD}\epsilon^{ijk}e_{Bi}e_{Cj}e_{Dk} \tag{62}$$

is indicated, and

$$\begin{aligned}-T^{Ai}&=-T^{Ai}_{(Y)}=\frac{1}{2}\tau^{A}{}_{jk}\,\epsilon^{ijk}\\&=\left(\partial_j e^{A}{}_{k}+\omega^{A}{}_{Bj}e^{B}{}_{k}\right)\epsilon^{ijk}\approx 0\,,\end{aligned} \tag{63}$$

$$\begin{aligned}-C^{ABi}&=-C^{ABi}_{(X)}\\&=2\,\mathbf{a}\Pi^{ABi}+R^{ABi}+\mathbf{c}e^{ABi}\approx 0\,.\end{aligned} \tag{64}$$

We made use of the convenient abbreviation (already suggested by (53))

$$W^{ABi}:=\frac{\epsilon^{ijk}}{2}\,W^{AB}{}_{jk} \tag{65}$$

for any two-form W^{AB}. With that definition we have e.g. from (53) and (54)

$$X^{ABi}=\Pi^{ABi},\quad Y^{Ai}=\pi^{Ai}\,. \tag{66}$$

By counting the degrees of freedom[12] one immediately infers that these $6+4+12+18=40$ constraints cannot all be independent first class for the 30 variables (e_{Ai},ω_{ABi}), and thus their Poisson bracket algebra will not close.

The Poisson brackets[13] of the six constraints Ω^{AB}, the constraints conjugate to ω_{AB0}, with all secondary constraints become (e.g. $E^{A'}$ is a shorthand for $E^{A'}\left(x^0,x^{i'}\right)$), $\delta:=\delta^3\left(x^i-x^{i'}\right)$

$$\{\Omega^{AB},E^{A'}\}=\left(\eta^{AA'}E^{B}-\eta^{BA'}E^{A}\right)\delta\,, \tag{67}$$

$$\begin{aligned}\{\Omega^{AB},\Omega^{A'B'}\}=&\left(\eta^{AA'}\Omega^{BB'}-\eta^{BA'}\Omega^{AB'}\right.\\&\left.+\eta^{BB'}\Omega^{AA'}-\eta^{AB'}\Omega^{BA'}\right)\delta\,,\end{aligned} \tag{68}$$

$$\begin{aligned}\{\Omega^{AB},C^{A'B'i'}\}=&\left(\eta^{AA'}C^{BB'i'}-\eta^{BA'}C^{AB'i'}\right.\\&\left.+\eta^{BB'}C^{AA'i'}-\eta^{AB'}C^{BA'i'}\right)\delta\,,\end{aligned} \tag{69}$$

$$\{\Omega^{AB},T^{A'i'}\}=\left(\eta^{AA'}T^{Bi'}-\eta^{BA'}T^{Ai'}\right)\delta\,, \tag{70}$$

and whence are weakly equal to zero. Equations (67)–(70) suggest that Ω^{AB} are proper generators of local Lorentz transformations.

The computation of the Poisson brackets of the 34 remaining, non-first class secondary constraints, on the one hand, yields

$$\{C^{ABi},C^{A'B'i'}\}=0\,, \tag{71}$$

$$\{T^{Ai},T^{A'i'}\}=0\,, \tag{72}$$

$$\{E^{A},E^{A'}\}=-\left(\mathbf{c}C^{AA'}+\Lambda\epsilon^{AA'}{}_{CD}T^{Ci}e^{D}{}_{i}\right)\delta\,, \tag{73}$$

which also weakly vanish. But, on the other hand, upon conservation of the secondary constraints $E^{A},\Omega^{AB},C^{ABi},T^{ABi}$ the remaining brackets

$$\{C^{ABi},T^{A'i'}\}=2\mathbf{a}\left(\eta^{AA'}e^{B}{}_{k'}-\eta^{BA'}e^{A}{}_{k'}\right)\epsilon^{i'ik'}\delta\,, \tag{74}$$

$$\begin{aligned}\{E^{A},T^{A'i'}\}=&\left(C^{AA'i'}+2\mathbf{a}\,\Pi^{AA'i'}\right.\\&\left.+3c_2\mathbf{j}\,e^{AA'i'}\right)\delta\,,\end{aligned} \tag{75}$$

$$\{E^{A},C^{A'B'i'}\}=\left(\mathbf{c}\left(\eta^{AA'}T^{B'i'}-\eta^{B'A}T^{A'i'}\right)\right.$$

[12] Our construction of the theory ensures that we have two degrees of freedom as ordinary Einstein–de Sitter GR (see Sect. 4.3 below).

[13] The appropriate formulas of Appendix C may be consulted.

$$+2\mathbf{a}\left(\eta^{AA'}\pi^{B'i'}-\eta^{B'A}\pi^{A'i'}\right)\right)\delta\,, \quad (76)$$

imply the appearance of ternary constraints and the fact that some of the secondary ones are second class. It should be noted that in two dimensional gravity (4) [23] the Poisson bracket algebra of the secondary constraints yields first class ones only, i.e. brackets like (74)–(76) do not appear there.

4.2 Ternary constraints

As a consequence of the Poisson brackets (75) and (76) the time evolution of the constraints E^A is given by

$$\begin{aligned}\dot{E}^A &\approx y_{A'i'}\left(2\mathbf{a}\,\Pi^{AA'i'}+3c_2\mathbf{j}\,e^{AA'i'}\right)\\&\quad+x_{A'B'i'}\mathbf{a}\left(\eta^{AA'}\pi^{B'i'}-\eta^{AB'}\pi^{A'i'}\right)\\&\approx 2\mathbf{a}\left(y_{A'i'}\Pi^{AA'i'}+x^A{}_{B'i'}\,\pi^{B'i'}\right)+3c_2\,\mathbf{j}y_{A'i'}\,e^{AA'i'}\\&\approx 0\,, \end{aligned}\quad (77)$$

and the time evolution of C^{ABi} follows from (76) and (74) as

$$\begin{aligned}\dot{C}^{ABi} &\approx y_{A'i'}2\mathbf{a}\left(\eta^{AA'}\,e^B{}_{k'}-\eta^{BA'}\,e^A{}_{k'}\right)\epsilon^{ii'k'}\\&\quad+e_{A'0}\,2\mathbf{a}\left(\eta^{AA'}\pi^{Bi}-\eta^{BA'}\pi^{A'i}\right)\\&\approx 2\mathbf{a}\left(\epsilon^{ii'k'}\left(y^A{}_{i'}\,e^B{}_{k'}-y^B{}_{i'}\,e^A{}_{k'}\right)\right.\\&\qquad\left.+e^A{}_0\pi^{Bi}-e^B{}_0\pi^{Ai}\right)\\&\approx 0\,. \end{aligned}\quad (78)$$

Finally the evolution equation for the twelve T^{Ai} are found from (74) and (75):

$$\begin{aligned}\dot{T}^{Ai} &\approx x_{A'B'i'}\,\mathbf{a}\left(\eta^{AA'}\,e^{B'}{}_{k'}-\eta^{AB'}\,e^{A'}{}_{k'}\right)\epsilon^{ii'k'}\\&\quad+\left(2\mathbf{a}\,\Pi^{AA'i}+3c_2\,\mathbf{j}\,e^{AA'i'}\right)e_{A'0}\\&\approx 2\mathbf{a}\left(x^{AB'}{}_{i'}\,e_{B'k'}\,\epsilon^{ii'k'}+\Pi^{AA'i}e_{A'0}\right)\\&\quad+3c_2\mathbf{j}\,e^{AA'i'}e_{A'0}\\&\approx 0\,. \end{aligned}\quad (79)$$

4.2.1 A relation between constraints

In order to render the system of equations consistent, one either has to determine certain Langrange multipliers or to introduce new constraints. We will see that both steps will be necessary for our system.

Performing the same calculation as the one leading to the second Bianchi identity[14]

$$\begin{aligned}\mathrm{D}R^{AB}|_{\partial_{123}} &\equiv \mathcal{D}_iR^{ABi}\\&= \partial_iR^{ABi}+\omega^A{}_{Li}R^{LBi}-\omega^B{}_{Li}R^{LAi}=0\end{aligned}$$

[14] Spatial indices are raised according to convention (65).

for the curvature with respect to the spatial components ∂_{123} applied to the constraint C^{ABi}, (64), yields

$$-\mathcal{D}_iC^{ABi}=2\mathbf{a}\mathcal{D}_i\Pi^{ABi}+\mathbf{c}\mathcal{D}_ie^{ABi}\,. \quad (80)$$

Solving (cf. (58), (59), (64) and (66))

$$\begin{aligned}-\Omega^{AB} &= -\Omega^{AB}_{(X)}-\Omega^{AB}_{(Y)}\\&= \mathcal{D}_i\Pi^{ABi}+\pi^{Ai}e^B{}_i-\pi^{Bi}e^A{}_i\,, \end{aligned}\quad (81)$$

for $\mathcal{D}_i\Pi^{ABi}$ and using the identity (10) for $S^{AB}=0$ one finds the six relations

$$\begin{aligned}\mathcal{D}_iC^{ABi} &= 2\mathbf{a}\left(\Omega^{AB}+\pi^{Ai}e^B{}_i-\pi^{Bi}e^A{}_i\right)\\&\quad+\mathbf{c}\left(T^{Ai}\,e^B{}_i-T^{Bi}\,e^A{}_i\right)\end{aligned}\quad (82)$$

between the constraints C^{ABi}, Ω^{AB} and T^{Ai}.

From (82) for invertible $\mathbf{a}$ follow the six weak relations (cf. (59))

$$J^{AB}:=-\Omega^{AB}_{(Y)}=\pi^{Ai}e^B{}_i-\pi^{Bi}e^A{}_i\approx 0 \quad (83)$$

between the 12 momenta π^{Ai} and their conjugate variables e_{Ai}. These relations exactly coincide with the constraints J^{AB} in the previous literature [12,13]. Although the Ω^{AB} appear naturally in our approach as the generators of the local Lorentz transformations (and vanish weakly by themselves), they may be tied as well by $\Omega^{AB}\approx -J^{AB}$ – as can be observed from (82) – to these constraints which played that role in the usual tetrad gravity. However, in a theory where ω_{ABi} and e_{Ai} are treated as independent variables as in our approach these are not the generators of the local Lorentz transformations, but only a part thereof. This can be seen, for instance, by looking at the Poisson brackets (C.9) and (C.10) of Appendix C where the "non-Lorentzian" parts just cancel in the sum of those two contributions to $\{\Omega^{AB},T^{A'i'}\}$.

4.2.2 Lagrange multipliers and ternary constraints

The key to the solution of the constraint equation (78) is the observation that the terms in the parentheses are just (cf. (66))[15]

$$Y^A\wedge e^B-Y^B\wedge e^A|^i\approx 0\,, \quad (84)$$

and (83) is nothing else but

$$Y^A\wedge e^B-Y^B\wedge e^A|_{123}\approx 0\,. \quad (85)$$

Thus one has to find a solution to

$$Y^A\wedge e^B-Y^B\wedge e^A\approx 0 \quad (86)$$

[15] A suggestive notation for the projections $\partial_{0jk}\,\frac{\epsilon^{ijk}}{2}=|^i$ and $\partial_{123}=|_{123}$ will be used from now for the related components of two-form and three-form equations.

which we already know to be uniquely $Y^A \approx 0$ (cf. lemma A of Appendix A):

$$y_{Ai} := Y_{A0i} \approx 0\,, \tag{87}$$

$$\pi^{Ai} := \frac{1}{2}\epsilon^{ijk} Y_{Ajk} \approx 0\,. \tag{88}$$

Equation (87) is a condition on Lagrange multipliers, whereas (88) are new (ternary) constraints. Actually, the result (87) and (88) was to be expected by our analysis of the Lagrangian equations of motion (cf. (47)).

By means of these two conditions, however, not only $\dot{C}^{ABi} \approx 0$ (cf. (78)), but also $\dot{E}^A \approx 0$ (cf. (77)) is fulfilled automatically.

Still, the conservation of the torsion constraint T^{Ai} (79) and the one of the new momenta constraints (88) (cf. (94) below) are not ensured yet. This will provide conditions on the remaining Lagrange mulipliers x_{ABi}.

4.2.3 Time evolution of π^{Ai}

The Poisson brackets of the ternary constraints (88) with all constraints are

$$\{\pi^{Ai}, E^{A'}\} = \left(\mathbf{c}\,\Pi^{AA'i} + \tilde{\Lambda}\mathbf{j}\,e^{AA'i}\right)\delta \tag{89}$$

$$\{\Omega^{AB}, \pi^{A'i'}\} = \left(\eta^{AA'}\,\pi^{Bi'} - \eta^{BA'}\,\pi^{Ai'}\right)\delta\,, \tag{90}$$

$$\{\pi^{Ai}, C^{A'B'i'}\} = \mathbf{c}\,\left(\eta^{AA'}e^{B'}{}_{k'} - \eta^{AB'}e^{A'}{}_{k'}\right)\epsilon^{ii'k'}\,\delta\,, \tag{91}$$

$$\{\pi^{Ai}, T^{A'i'}\} = -\eta^{AA'}\,\epsilon^{ii'k'}\,\delta_{,k'} - \omega^{AA'}{}_{k'}\,\epsilon^{ii'k'}\,\delta\,, \tag{92}$$

$$\{\pi^{Ai}, \pi^{A'i'}\} = 0\,. \tag{93}$$

Taking into account (87) and (88) their time evolution is found to be

$$\begin{aligned}\dot{\pi}^{Ai} &\approx e_{A'0}\left(\mathbf{c}\,\Pi^{AA'i} + \tilde{\Lambda}\mathbf{j}\,e^{AA'i}\right)\\ &\quad + \frac{1}{2}\,x_{A'B'i'}\mathbf{c}\,\left(\eta^{AA'}e^{B'}{}_{k'} - \eta^{AB'}e^{A'}{}_{k'}\right)\epsilon^{ii'k'}\\ &\approx \mathbf{c}\left(e_{A'0}\Pi^{AA'i} + e_{B'k'}\,x^{AB'}{}_{i'}\,\epsilon^{ii'k'}\right) + \tilde{\Lambda}\mathbf{j}\,e_{A'0}e^{AA'i}\,.\end{aligned} \tag{94}$$

It should be noted that the right-hand side of (92) is the only instance where a derivative of the delta functional occurs in our approach. However, since the corresponding Lagrange multipliers y_{Ai} of the torsion constraints T^{Ai} vanish anyhow by (87) and thus could be dropped in (94), these particular Poisson brackets are of no importance for the Hamiltonian analysis.

4.3 The remaining Lagrange multipliers x_{ABi}

4.3.1 Reformulation of the problem

Defining a two-form

$$Z^{AB} \equiv 2\mathbf{a}X^{AB} + \left(\mathbf{c} - \frac{\Lambda}{6}\right)e^{AB}\,, \tag{95}$$

we show in this subsection that the solution of $\dot{T}^{Ai} \approx 0$ (79) and $\dot{\pi}^{Ai} \approx 0$ (94) can be reformulated as the solution of the two weak three-form equations

$$Z^{AB} \wedge e_B \approx 0\,, \tag{96}$$

$$\widehat{Z^{AB}} \wedge e_B = \mathbf{j}\,Z^{AB} \wedge e_B \approx 0\,. \tag{97}$$

The components of X^{AB} in (95) consist of the Lagrange multipliers $X_{AB0i} = x_{ABi}$ and of the $X^{AB}{}_{jk}$ which are proportional to the momenta conjugate to $\omega^{AB}{}_i$ (cf. (53)). Therefore, (96) and (97) will be inhomogeneous linear equations for x_{ABi} whose solution will be discussed in Sect. 4.3.2.

The derivation of each of (96) and (97) consists of separate proofs for the projections $|^i$ and $|_{123}$ (cf. footnote 15).

We first note that the bracket in the second line of (79) is nothing else than $X^{AB} \wedge e_B|^i$. Adding a vanishing term $(c_0 - \Lambda/6)\,e^{AB} \wedge e_B$ to the last term of (79), with $c_0 + \mathbf{j}c_2 = \mathbf{c}$ allows one to identify that term with the component $|^i$ of the last two terms (95). This proves (96)$|^i$. Also in (94) the bracket in the second line is $X^{AB} \wedge e_B\,|^i$. Together with the prefactor $\mathbf{c} = \left(\mathbf{c}\mathbf{a}^{-1}\right)\,\mathbf{a} = -4\,(\gamma + \mathbf{j})\,\mathbf{a}$ (cf. (33)) the combination $2\mathbf{a}X^{AB} \wedge e_B\,|^i$ can be expressed by means of (the already proved) (96)$|^i$ by the corresponding component from the last two terms in (95). Again we may add a vanishing term proportional $c_2\,e^{AB} \wedge e_B$ in order to produce the combination $c_0 + \gamma c_2$ in (36). Together with (37) this leads (up to an overall factor) to (97)$|^i$.

To show the validity of (96)$|_{123}$ and (97)$|_{123}$ the constraints (57), (63), (64) and (88) suffice. Indeed from (57) with (88) one arrives at the same expression as the one in (94) but projected upon $|_{123}$ instead of $|^i$. However, before concluding that (97)$|_{123}$ holds, according to the intermediate step of our argument in the case $|^i$, we first need the proof of (96)$|_{123}$. Multiplying the constraint (64) with e_{Bi} one realizes immediately that the result in the expression would coincide with (96)$|_{123}$ if $R^{ABi} \wedge e_{Bi} \approx 0$. Taking the definition of torsion from the torsion constraint (63) and R^{AB} from (2) the second Bianchi identity in the form

$$\partial_i T^{Ai} + \frac{\epsilon^{ijk}}{2}\,\omega^{AD}_j\,T^k_B = R^{ABi} e_{Bi}\,, \tag{98}$$

of course, also holds here which for (63) implies the desired result. Thus (96) and (97) are valid for all components.

4.3.2 Solution of x_{ABi}

Each one of the two equations (96) and (97) has 12 components $|^i$ and four components $|_{123}$, yielding altogether 32 linear equations for the 18 Lagrangian multipliers x_{ABi}, eight of which are identities not including x_{ABi}. Therefore, six relations between those components should hold to make a unique solution possible.

As a first step one has to solve (96) and (97) as a system of linear inhomogeneous equations for Z^{AB}_{0i}, where the RHS is linear in $e^A{}_0$. Then from the definition (95) the solution for $x^{AB}{}_i$ can be read off directly. This first step is by no means straightforward, however, it is enormously simplified by exploiting the invariance of (96) and (97) which allows one to perform that calculation in a suitable frame. Locally the components ${e^A}_I$ of the vierbeins can be transformed by a Lorentz transformation ($\gamma = (1 - v^2)^{-\frac{1}{2}}, v^2 = v_a v^a$)

$$ {\ell^A}_B = \begin{pmatrix} \gamma & \gamma\, v_b \\ \gamma v^a & \left[\delta^a_b + (\gamma - 1)\ \frac{v^a v_b}{v^2}\right] \end{pmatrix}^A_{\ \ B} \tag{99} $$

and a diffeomorphism restricted to the space components e^a_i

$$ {t_J}^I = \begin{pmatrix} 1 & 0 \\ 0 & {t_j}^i \end{pmatrix}, \tag{100} $$

such that

$$ {\tilde{\tilde{e}}^A}{}_J = {t_J}^I\, \tilde{e}^A_I = {t_J}^I\, {\ell^A}_B\, {e^B}_I \tag{101} $$

is brought into the form

$$ {\tilde{\tilde{e}}^0}{}_i = 0\,, \tag{102} $$

$$ {\tilde{\tilde{e}}^a}{}_i = \delta^a_i\,. \tag{103} $$

Actually ${t_j}^i$ is nothing else than the 3D inverse ${}^3{\tilde{E}_a}{}^i$ (in the sense ${}^3{\tilde{E}_a}{}^i\, {\tilde{e}^b}{}_i = \delta^b_a$) where the index a is replaced by j. Physically (102) means that at the space time point considered, $\tilde{e}^0_i$, the analogon of the "shift", is set to zero and thus transformed to the rest frame, whereas the directions of the 3D holonomic frame have been rotated into the Lorentz directions renormalized to one.[16] Equation (102) with (99) implies the velocity

$$ v_b = -{}^3{E_b}^i\, {e^0}_i, \tag{104} $$

where now the 3D inverse of e^a_j (in the sense ${}^3{E_b}^i\, {e^b}_j = \delta^i_j$) has been used.

The condition $v^2 < 1$ for the local Lorentz transformation amounts to the requirement ${}^3g^{ij} {e^0}_i\, {e^0}_j < 1$ for the norm of e^0_i with respect to the local 3D metric. Light-like boosts are excluded in our present paper.

Clearly ${\ell^A}_B$, as well as its inverted form $\ell(-v)$ which appears, when one returns to the original frame, will depend in a complicated (non-polynomial) way on the original ${e^0}_i$ and ${e^a}_i$.

But the inverse ${\left(t^{-1}\right)_i}^j$ simply coincides with $\tilde{e}^{(a)}_i$, where, as in the inverse ${}^3{\tilde{E}_a}{}^i$ above, the index (a) is replaced by the index j:[17]

$$ \tilde{Z}^{ab}_{0i} = \tilde{e}^{(j)}_i\, \tilde{\tilde{Z}}^{ab}_{0(j)} = {\ell^{(j)}}_B\, {e^B}_i\, \tilde{\tilde{Z}}^{ab}_{0(j)}, \tag{105} $$

$$ \tilde{Z}^{ab}_{0i} = \tilde{e}^{(j)}_i\, \tilde{\tilde{Z}}^{0b}_{0(j)} = {\ell^{(j)}}_B\, {e^B}_i\, \tilde{\tilde{Z}}^{0b}_{0(j)}. \tag{106} $$

[16] This implies that in the present paper we restrict for simplicity to e^A_I which are not light-like, a case which must be treated separately.

[17] This mixed usage of some indices like j now is indicated by (j).

We recall that the Lorentz transformations in (105) and (106) depend on (minus) the velocity (104). The components of Z in the general frame (without tilde)

$$ Z^{AB}_{0j} = {\left(\ell^{-1}\right)^A}_C\, {\left(\ell^{-1}\right)^B}_D\, \tilde{Z}^{CD}_{0j} \tag{107} $$

contain two more Lorentz transformations with (minus) the velocity (104). Therefore, at the end of the day, the factors in the relation between Z and $\tilde{\tilde{Z}}$ are linear in e^B_i, except for the Lorentz transformations of the type (99) with (non-linear) velocity (104) and, of course, a non-polynomial dependence on v in γ.

For foliations which do not require large velocities, a low velocity limit of (99) suffices. However, (102) cannot be reached in this way by a Galilei transformation. Instead for small velocities $v \ll 1$ also a transformation

$$ \tilde{\ell}^A_B = \begin{pmatrix} 1 & v^a \\ 0 & \delta^a_b \end{pmatrix}^A_{\ \ B}, \tag{108} $$

may be considered which is a group contracted [34] Lorentz transformation, obtained by rescaling $v^a \to \lambda v^a$, $x^a \to \lambda^{-1} x^a$ in the limit $\lambda = 0$. This is a "twisted" version of the more familiar contraction towards the Galilei transformation ($v^a \to \lambda v^a$, $x^0 \to \lambda^{-1} x^0$).

Having shifted essential parts of the problem into ℓ^a_b and t the solution in the frame (102) and (103) is relatively simple. Details are given in Appendix D (in the frame (102) and (103) holonomic and anholonomic spatial indices may be raised, lowered and combined freely):

$$ \tilde{\tilde{Z}}^{0a}_{0i} = -\frac{1}{4} {\epsilon^{aB}}_{CD}\, \tilde{\tilde{Z}}^{CD}_{jk}\, \tilde{\tilde{e}}_{B0}\, \epsilon^{ijk}, \tag{109} $$

$$ \tilde{\tilde{Z}}^{ab}_{0i} = -\frac{1}{2}\, \epsilon^{abj}\, \tilde{\tilde{Z}}^{iB}_{rs}\, \epsilon^{jrs}\, \tilde{\tilde{e}}_{B0}. \tag{110} $$

Together with the vanishing multipliers (87) this resolves the problem. As expected, the solution is found to be the consequence of 18 among the 32 equations (96) and (97), the remaining ones just yielding identities. The expressions on the RHS of (109) and (110) not only are linear in $\tilde{\tilde{e}}^a_0$ but also in the canonical momenta $\tilde{\tilde{\Pi}}^{ABi}$ (cf. (66)) proportional to $\tilde{\tilde{Z}}^{ABi}_{jk}$. These properties survive the (only ${e^A}_i$-dependent) linear transformations t^{-1}, $\ell(-v)$ when returning by (108)–(110) to the original frame e^A_I.

Thus the Lagrange multipliers are expressible as

$$ x_{ABi} = {h_{ABi}}^C\, e_{C0}, \tag{111} $$

where the coefficients ${h_{ABi}}^C = -{h_{BAi}}^C$ are non-polynomial functions of ${e^A}_i$, but still linear in Π^{ABi}. According to (55) with (87) and (111) this yields a Hamiltonian of the form (cf. also the last [13])

$$ H = \int \mathrm{d}^3x \left(e_{A0} \tilde{E}^A + \frac{1}{2} \omega_{AB0} \Omega^{AB} \right), \tag{112} $$

with the new constraints

$$\tilde{E}^A = E^A + \frac{1}{2}\, h_{CDi}{}^A\, C^{CDi}\,, \tag{113}$$

which must be first class by comparing with the number of degrees of freedom: Our theory involves the 30 phase space variables (e_{Ai}, ω_{ABi}), the $N_\mathrm{f} = 10$ first class constraints $\left(\tilde{E}^A, \Omega^{AB}\right)$ or equivalently the 10 independent Lagrange multipliers (e_{A0}, ω_{AB0}), the $12+18+12 = 42$ second class constraints $\left(T^{Ai}, C^{ABi}, \pi^{Ai}\right)$ with the six relations (82) between them, giving $N_\mathrm{s} = 36$ independent second class constraints. Thus, we find the number of physical degrees of freedom according to standard textbook lore (cf. e.g. [35]) as

$$N_\mathrm{df} = N_\mathrm{ph} - N_\mathrm{f} - \frac{1}{2}N_\mathrm{s} = 30 - 10 - \frac{1}{2}(42-6) = 2\,, \tag{114}$$

which is precisely the correct one for GR and thus agrees with counting according to our Hamiltonian analysis.

It should be noted that independent of the specific variables one always encounters 10 first class constraints: In Einstein–Cartan gravity with 12 variables e_{Ai} one obtains $12 - 10 = 2$ degrees of freedom, in the Ashtekar approach with 18 ω_{ABi} the counting is $18 - 10 - \frac{12}{2} = 2$, where the last term originates from the second class constraints in that formulation.

The algebra of first class constraints (with respect to appropriate Dirac brackets) should (strongly) equal the Poincaré algebra:[18]

$$\{\Omega^{AB}, \Omega^{A'B'}\} \tag{115}$$
$$= \left(\eta^{AA'}\Omega^{BB'} - \eta^{BA'}\Omega^{AB'} + \eta^{BB'}\Omega^{AA'} - \eta^{AB'}\Omega^{BA'}\right)\delta\,,$$

$$\{\Omega^{AB}, \tilde{E}^{A'}\} = \left(\eta^{AA'}\tilde{E}^B - \eta^{A'B}\tilde{E}^A\right)\delta\,, \tag{116}$$

$$\{\tilde{E}^A, \tilde{E}^{A'}\} = 0\,. \tag{117}$$

So far this final check has not been made, because of the very messy algebra.

5 Conclusion and outlook

Motivated by the success of an analogous program in $1+1$ dimensional gravity theories we present a reformulation of Einstein (–de Sitter) gravity, strictly in terms of Cartan variables (one-form vierbeins e_A and spin connection ω_{AB}), and of a first order derivative Hamiltonian action. We do not rely upon the Palatini mechanism to arrive at the dependent spin connection, but keep a separate condition for vanishing torsion. For that and in order to reproduce the usual Einstein gravity, sets of auxiliary two-form fields (X^{AB}, Y^A) are introduced. Part of them become the momenta, canonically conjugate to the dynamical components of (space-like parts) of the spin connection and of the vierbeins. The remaining components of the auxiliary fields represent Lagrangian multipliers.

By comparison with the situation in $1+1$ dimensional gravity, resulting from spherical reduction of Einstein gravity in four dimensions, the dilaton field X in that case may even have a counterpart among certain "dilaton fields" contained in X^{AB} which also here may be the result of a compactification mechanism from gravity in dimensions larger than four.

Another speculation suggests itself by the different ways the two contributions to the effective cosmological constant Λ in (37) appear in the present formulation. By now it has been established by a host of astronomical data [38] that Λ, the dark energy, has a small but positive value. This disagrees with supergravity which requires an anti-de Sitter space with negative Λ [39]. That this result is unchanged in related dilaton supergravity, e.g. for the $1+1$ dimensional case has been demonstrated in [40]. Assuming that the parts (23) and (24) of the action by themselves are the result of compactifying a higher dimensional supergravity, whereas the contribution (25) is determined by the positive contribution from the standard model, there is a chance of compensation, albeit still suffering from the usual tremendous fine-tuning problem. In any case, the undetermined sign of the second term in (37) allows for such a solution.

In our approach we do not follow the usual route of an ADM decomposition, instead being led rather naturally to a Hamiltonian analysis, reminiscent of the one in non-abelian gauge theory. Nevertheless, our approach completely differs from the one advocated by Ashtekar [5] and the work which has developed from that. On the other hand, in the course of our analysis we encounter constraints which in classical works on the interpretation of the e.g. Lorentz invariance had been introduced – to a varying degree – as ad hoc conditions (cf. e.g. [12,15]).

The price we have to pay is the appearance of ternary and second class constraints which determines the aforementioned Lagrange multipliers, the ternary constraints being nothing else than the momenta canonically conjugate to the space components of the vierbeins. Already in order to arrive at this point required the evaluation of straightforward but – at intermediate steps – very lengthy formulas for Poisson brackets which has been done by hand and verified on the computer by a program package written for this purpose.

The very symmetric structure of our approach allowed us to formulate the final Hamiltonian in a manner akin to non-abelian gauge theories, i.e. in terms of first class constraints (with respect to appropriate Dirac brackets), multiplied by "time" components (e_{A0}) of the vierbeins and of the spin connection (ω_{AB0}) – just like the Gauss con-

[18] Of course, relations of this form, with the exception of (117), have been conjectured from symmetry arguments for a long time. They are e.g. the basis of Poincaré gauge theory [11,36,37] (cf. also the last reference [13]). Primary constraints with such an algebra also appeared in [29] in a tetrad theory with dependent spin connection. However, in contrast to the results of those works, as seen from (90) our Lorentz constraints possess *all* the correct commutators, especially also the ones with the momenta π_i^A of the vierbeins.

straints in Yang–Mills theories. The explicit check of the first class property for the set Ω^{AB}, $\tilde{E}^A$ still requires another exacting mathematical effort, but we are convinced that the constraints of the (local) Poincaré algebra (115)–(117) should be reproduced directly.

At this point we should emphasize again that our approach contains the original $SO(3,1)$ local Lorentz symmetry with associated connection – not reduced to $SO(3,\mathbb{C})$ as in Ashtekar's formulation. In comparison with other manifest $SO(3,1)$ approaches (cf. e.g. [30,31]) our constraints and the associated constraint algebra are much less involved.

Our approach lends itself to generalizations[19] in several directions, the most obvious one being to dynamical torsion which also plays a role in the attempts to make GR renormalizable (cf. e.g. [42]). Then an ansatz like (23)–(25) must be supplemented by further terms linear and quadratic in Y^A. Dynamical torsion is a natural consequence of Poincaré gauge theories of gravity [11,36,37].[20]

Another interesting field of applications may be the teleparallelism formulations of Einstein gravity where the spin connection is flat and the whole dynamical structure resides in the Weitzenböck connection which is determined by torsion alone [44] (cf. also here the review [43]).

However, it should be kept in mind that a polynomial ansatz with auxiliary fields as in our case places certain restrictions upon quadratic terms in torsion and – for that matter – also upon terms quadratic in the curvature tensor (cf. the remarks after (15) and (16)). On the other hand, the extension of the space of variables by auxiliary fields may be helpful in a quantum theory. Of course, it cannot be expected that a full background independent quantization of gravity can be obtained as in $1+1$ dimensions [22, 23]. But our approach, although different from Ashtekar's one [5], also may allow for some of the developments which were made possible by its structural similarity to Yang–Mills theory in the Hamiltonian analysis.

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Appendix A: Form equations

In the analysis of the four dimensional Hamiltonian linear derivative action one frequently encounters three-form equations of the type

$$Z^A \wedge e^B - Z^B \wedge e^A = S^{AB} \quad A, B = 0, ..., 3 \tag{A.1}$$

for the two-form components (Z^A) of a $(1,2)$ bi-form[21] for the components $\left(S^{AB}\right)$ of an arbitrary right-hand side (two, three)-form.

Lemma A. Equations (A.1) have a unique solution (cf. Appendix A of [45])

$$Z^A = \iota_K S^{AK} - \frac{1}{4}\,\iota_L \iota_K S^{KL} \wedge e^A\,, \tag{A.2}$$

with the inner derivative ι_K meaning evaluation of the form on the basis vector E_K.

In our present paper we mostly need the homogeneous case $S^{AB} = 0$. Here clearly one solution is the trivial one $Z^A = 0$. A direct proof for the uniqueness of this solution to (A.1) with $S^{AB} = 0$ follows from considering the equations for the components $Z^A{}_{CD}$ in $Z^A = Z^A{}_{CD}\, e^{CD}$ which follow from (A.1) at $S^{AB} = 0$:

$$Z^A{}_{CD} e^{CDB} = Z^B{}_{EF} e^{EFA} \quad A \neq B\,. \tag{A.3}$$

This gives for $A = 0, B = 1$ the three-form equation[22]

$$\begin{aligned} &Z^0{}_{23}e^{231} + Z^0{}_{02}e^{021} + Z^0{}_{03}e^{031} \\ &= Z^1{}_{23}e^{230} + Z^1{}_{12}e^{120} + Z^1{}_{13}e^{130}\,, \end{aligned} \tag{A.4}$$

from which one immediately infers $Z^0{}_{23} = 0, Z^1{}_{23} = 0$, as well as $\left(e^{021} = -e^{120}\right)$

$$Z^0{}_{02} = -Z^1{}_{12}\,, \tag{A.5}$$

$$Z^0{}_{03} = -Z^1{}_{13}\,. \tag{A.6}$$

One sees that they are of the form $Z^A{}_{AB} = -Z^C{}_{CB}$ for $A \neq C$. Thus considering all three equations ($A \neq B$ with $A, B = 0, ..., 3$) for the components $Z^A{}_{AB}$ one finally has (without loss of generality we may set e.g. $B = 2$)

$$Z^0{}_{02} = -Z^1{}_{12} = Z^3{}_{32} = -Z^0{}_{02} \Rightarrow Z^0{}_{02} = 0\,, \tag{A.7}$$

and thus the trivial solution $Z^A = 0$ is the unique one.

Appendix B: Hat calculus

If we consider the hat operator as acting not only on bi-forms but on any antisymmetric object X_{AB} with two $SO(3,1)$ indices, using the definition

$$\widehat{X_{AB}} := \frac{1}{2}\epsilon_{AB}{}^{CD} X_{CD}\,, \tag{B.1}$$

and raising and lowering indices with the Minkowski metric η_{AB} we find the following rules.

[19] For a comprehensive review of generalized gravity cf. [41]

[20] A review on torsion, covering in particular the literature on actions quadratic in torsion is [43].

[21] The notion of a bi-form proves to be very useful in mathematics on manifolds. In short, a bi-form is a tensor product of alternating forms and alternating vectors, i.e. alternating vectors whose components are forms. Thus, for example, a $(1,2)$ bi-form $Z = Z_A E^A$ is a vector which two-form components Z_A with respect to the basis E^A. Further useful abbreviations are $\left(e^2\right)^{AB} = e^{AB} = e^A \wedge e^B$, $\left(e^3\right)^{ABC} = e^{ABC} = e^A \wedge e^B \wedge e^C$ etc.

[22] Here for notational simplicity we deviate from our convention to underline special Lorentz components.

Lemma B (hat rules). Let $\diamond$ denote any bilinear operator and X_{AB}, Y_{AB} be any antisymmetric object with indices $A, B = 0, 1, 2, 3$; then the following identities hold:

$$\widehat{\widehat{X_{AB}}} = -X_{AB}\,, \tag{B.2}$$

$$\widehat{X_{KL}} \diamond Y^{KL} = X_{KL} \diamond \widehat{Y^{KL}}\,, \tag{B.3}$$

$$X_{AL}\widehat{\diamond Y^L}_B = \frac{1}{2}\left(X_{AL}\widehat{\diamond Y^L}_B - X_{BL}\widehat{\diamond Y^L}_A\right), \tag{B.4}$$

$$X_{AL}\widehat{\diamond Y^L}_B = \frac{1}{2}\left(\widehat{X_{AL}} \diamond Y^L{}_B - \widehat{X_{BL}} \diamond Y^L{}_A\right), \tag{B.5}$$

$$\widehat{X_{AL}} \diamond Y^L{}_B = -X_{BL} \diamond \widehat{Y^L{}_A}\,, \quad A \neq B\,, \tag{B.6}$$

$$\widehat{X_{AL}} \diamond \widehat{Y^L{}_B} = \frac{1}{2}\eta_{AB} X_{KL} \diamond Y^{KL} + X_{BL} \diamond Y^L{}_A\,. \tag{B.7}$$

Of course, this hat operation on operands with antisymmetric indices is again linear. Moreover it commutes with the exterior derivative because ϵ_{ABCD} and η_{AB} are merely numbers, yielding the identities

$$\mu X_{AB} \widehat{+ \nu} Y_{AB} = \mu \widehat{X_{AB}} + \nu \widehat{Y_{AB}}\,, \tag{B.8}$$

$$d\widehat{Z_{AB}} = \widehat{dZ_{AB}}\,. \tag{B.9}$$

Covariant derivatives on bi-vectors are defined by

$$(DX)^{AB} = dX^{AB} + \omega^A{}_C \wedge X^{CB} + \omega^B{}_C \wedge X^{AC}. \tag{B.10}$$

Appendix C: Poisson brackets

Our Poisson brackets are defined as

$$\{A\left(x^0, x^i\right), B\left(x^0, x'^i\right)\} \tag{C.1}$$
$$= \int \mathrm{d}^3\tilde{x} \left\{ \frac{\delta A}{\delta\,\tilde{e}_{Ai}} \frac{\delta B}{\delta \tilde{\Pi}^{Ai}} + \frac{1}{2} \frac{\delta A}{\delta \tilde{\omega}_{ABi}} \frac{\delta B}{\delta \tilde{\Pi}^{ABi}} - (A \leftrightarrow B) \right\}.$$

An expression like $\tilde{e}_{Ai}$ is the shorthand for $e_{Ai}\left(x^0, \tilde{x}^i\right)$ etc. Note the factor $\frac{1}{2}$ in the second term which permits independent summation of A and B in the antisymmetric components.

Another simple formula which is crucial in order to obtain brackets of constraints which do not produce derivatives of δ-functions, and in this sense resembling the ones in Yang–Mills theory, is

$$\int_{x'} \left[f(x')\, \delta(x'' - x)\, \partial' \delta(x' - x'') + (x' \leftrightarrow x) \right]$$
$$= (\partial f)\, \delta(x' - x)\,. \tag{C.2}$$

In order to give an idea of the brackets to be evaluated, a few examples are given below for different portions of Ω^{AB} and E^A, as defined in (58)–(62). In all cases the terms in the bracket again are taken at $\left(x^0, x^i\right)$ and $\left(x^0, x'^i\right)$, respectively; $\delta\left(x^i - x'^i\right) = \delta,\ \delta_{,i} = \partial_i\,\delta$:

$$\left\{\Omega^{AB}_{(X)}, a\Pi^{A'B'i'}\right\}$$
$$= a\left(\eta^{AA'}\Pi^{BB'i'} - \eta^{BA'}\Pi^{AB'i'} + \eta^{BB'}\Pi^{AA'i'} - \eta^{AB'}\Pi^{B'A'i'}\right)\delta, \tag{C.3}$$

$$\left\{\Omega^{AB}_{(X)}, R^{A'B'i'}\right\}$$
$$= -\left(\eta^{AA'}R^{BB'i'} - \eta^{BA'}R^{AB'i'} + \eta^{BB'}R^{AA'i'} - \eta^{AB'}R^{BA'i'}\right)\delta, \tag{C.4}$$

$$\left\{\Omega^{AB}_{(X)}, E^{A'}_{(X)}\right\} \tag{C.5}$$
$$= \left(\eta^{AA'}E^B - \eta^{BA'}E^B\right)\delta + c\left(e^A{}_i\,\Pi^{BA'i} - e^B{}_i\,\Pi^{AA'i}\right)\delta,$$

$$\left\{\Omega^{AB}_{(X)}, E^{A'}_{(Y)}\right\}$$
$$= \left(\eta^{BA'}\pi^{Ai} - \eta^{AA'}\pi^{Bi}\right)\delta_{,i} + \left(\eta^{BA'}E^A_{(Y)} - \eta^{AA'}E^B_{(Y)}\right)\delta + \left(\omega^{BA'}{}_i\pi^{Ai} - \omega^{AA'}{}_i\pi^{Bi}\right)\delta, \tag{C.6}$$

$$\left\{\Omega^{AB}_{(Y)}, E^{A'}_{(X)}\right\}$$
$$= c\left(\Pi^{AA'i}e^B{}_i - c\Pi^{BA'i}e^A{}_i\right)\delta, \tag{C.7}$$

$$\left\{\Omega^{AB}_{(Y)}, E^{A'}_{(X)}\right\} \tag{C.8}$$
$$= \left(\omega^{AA'}{}_i\pi^{Bi} - \omega^{BA'}{}_i\pi^{Ai}\right)\delta + \left(\eta^{AA'}\pi^{Bi} - \eta^{BA'}\pi^{Ai}\right)\delta_{,i},$$

$$\{\Omega^{AB}_{(X)}, T^{A'i'}\}$$
$$= \left(\eta^{AA'}T^{Bi'} - \eta^{BA'}T^{Ai'}\right)\delta + \left(\eta^{BA'}e^A{}_{k'} - \eta^{AA'}e^B{}_{k'}\right)\epsilon^{i'j'k'}\,\delta_{,j'} + \left(\omega^{BA'}{}_{j'}\,e^A{}_{k'} - \omega^{AA'}{}_{j'}\,e^B{}_{k'}\right)\epsilon^{i'j'k'}\,\delta, \tag{C.9}$$

$$\{\Omega^{AB}_{(Y)}, T^{A'i'}\}$$
$$= \left(\eta^{AA'}e^B{}_{k'} - \eta^{BA'}e^A{}_{k'}\right)\epsilon^{i'j'k'}\,\delta_{,j'} + \left(\omega^{AA'}{}_{j'}\,e^B{}_{k'} - \omega^{BA'}{}_{j'}\,e^A{}_{k'}\right)\epsilon^{i'j'k'}\,\delta. \tag{C.10}$$

Appendix D: Solution for Lagrangian multipliers

The solution of (96) and (97) is most easily established in the frame (102) and (103), dropping double tildes in this appendix:

$$e^0{}_i = 0, \qquad e^a{}_i = \delta^a_i. \tag{D.1}$$

Projecting (96) and (97) upon ∂_{0ij}, respectively, yields 24 inhomogeneous equations for the 18 variables Z^{AB}_{0j}

$$Z^{AB}_{0j}\,e_{Bk}\,\epsilon^{ijk} = -\frac{1}{2}\,Z^{AB}_{jk}\,\epsilon^{ijk}\,e_{B0} \equiv r^{Ai}\,, \tag{D.2}$$

$$\hat{Z}^{AB}_{0j}\, e_{Bk}\, \epsilon^{ijk} = -\frac{1}{2}\, \hat{Z}^{AB}_{jk}\, \epsilon^{ijk}\, e_{B0} \equiv \hat{r}^{Ai}\,, \tag{D.3}$$

where the RHS are linear in the momenta Π^{ABi} (cf. (65), (66) and (95)) and in e^A_0.

The projections by ∂_{123} imply eight restrictions upon the 18 momenta canonically conjugate to ω_{iAB}:

$$Z^{AB}_{ij}\, e_{Bk}\, \epsilon^{ijk} = 0\,, \tag{D.4}$$
$$\hat{Z}^{AB}_{ij}\, e_{Bk}\, \epsilon^{ijk} = 0\,, \tag{D.5}$$

which is reasonable because for $\pi^{Ai}_e \approx 0$ the effective phase space should be of dimension 20.

Multiplying (D.4) and (D.5) by e_{A0} verifies two relations between the r^{Ai} and $\hat{r}^{Ai}$:

$$r^{Ai}\, e_{Ai} = 0, \tag{D.6}$$
$$\hat{r}^{Ai}\, e_{Ai} = 0. \tag{D.7}$$

In the frame (D.1) and (D.2) and (D.3) are each separated into the ones for $A = \underline{0}$ and $A = a$ (position and combinations of special indices here may be moved freely):

$$Z^{\underline{0}a}_{0j}\, \epsilon^{ija} = r^{\underline{0}i}\,, \tag{D.8}$$
$$\hat{Z}^{\underline{0}a}_{0j}\, \epsilon^{ija} = \hat{r}^{\underline{0}i}\,, \tag{D.9}$$

and the 18 equations for the 18 Z^{AB}_{0j}:

$$\left(M^{ij}\right)^a\, Z^{\underline{0}a}_{0j} = \hat{r}^{ai}\,, \tag{D.10}$$
$$-\left(M^{ij}\right)^a\, \hat{Z}^{\underline{0}a}_{0j} = r^{ai}\,, \tag{D.11}$$

where for symmetry reasons $\hat{Z}^{\underline{0}a} = \frac{1}{2}\epsilon^{\underline{0}a}{}_{bc}\, Z^{bc}$ have been introduced as a second triplet of the Z. Inspection of the matrices

$$\begin{aligned}
\left(M^{ij}\right)^{\underline{1}} &= \left(0 \quad \epsilon^{3ij} \quad -\epsilon^{2ij}\right),\\
\left(M^{ij}\right)^{\underline{2}} &= \left(-\epsilon^{3ij} \quad 0 \quad \epsilon^{1ij}\right),\\
\left(M^{ij}\right)^{\underline{3}} &= \left(\epsilon^{2ij} \quad -\epsilon^{1ij} \quad 0\right)
\end{aligned} \tag{D.12}$$

in (D.10) and (D.11) shows that Z^{ab}_{0j} for $b \neq i$ only have one entry in the matrix M, so that e.g. for (D.10) as long as $b \neq i$

$$Z^{ab}_{0i} = -\epsilon^{abj}\, \hat{r}^{\underline{ij}}\,. \tag{D.13}$$

On the other hand, for $(b) = (i)$ (this notation indicates no sum over those indices) e.g. from (D.2) the set of linear equations

$$Z^{\underline{0}(a)}_{0(a)} = \frac{1}{2}\, A^a{}_b\, \hat{r}^{\underline{bb}}\,, \tag{D.14}$$

$$A^a{}_b = \begin{array}{cc} & a \\ \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} & \\ & b \end{array} \tag{D.15}$$

follows. However, thanks to (D.7) for unit matrix e_{ai} (D.1) we may use $\hat{r}^{\underline{a}a} = 0$ so that (D.13) holds as well for equal indices $b = i$. Therefore, with double tilde restored (D.13) becomes (110) of Sect. 4.3.2, and (D.10) by the same steps with

$$Z^{\underline{0}a}_{0i} = \hat{r}^{ai} \tag{D.16}$$

leads to (109).

So far the searched for 18 variables have been determined from 18 equations. As expected from the rank of the system the still unused (D.8) and (D.10) indeed turn out to be satisfied identically.

In view of the complicated dependence of the velocity in (99) one could avoid that transformation in the transition from Z to $\tilde{\tilde{Z}}$ by dropping the first condition (D.1). The strategy of the solutions to (D.2) and (D.3) in principle remains the same as indicated above, but, of course, the solution of the linear system is more involved.

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